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KILPELAINEN-MALY TECHNIQUE FOR THE GENERAL CASE OF DIVERGENCE QUASILINEAR ELLIPTIC EQUATIONS

In the paper we prove an iteration lemma for the general case of quasilinear elliptic equations in divergence form:

$$-div\left(g(a(x),|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(x) \geq 0.$$

As in the case of p-Laplace operator, for which iteration lemma has been proved in first by Kilpelainen and Maly, our obtained result serves a basic instrument for further investigation of such type of quasilinear elliptic equations. With the help of mentioned lemma it is established the Harnack –type inequality for equations under consideration in terms of nonlinear Wolf potentials.

Keywords: quasilinear elliptic equations, iteration technique, second order partial differential equations, p-Laplace operator, Wolf potentials, double-phase equations, pointwise estimates.

Introduction

This paper is devoted to the extension of well-known Kilpelainen-Maly technique [1], which has been applied in first for p-Laplace operator and measure μ in right hand side of the equation

$$-\Delta_p u = \mu, \tag{1}$$

$$\Delta_p u := div(|\nabla u|^{p-2} \nabla u),\ p > 2.$$

Using proposed iteration technique, there has been proved [1] pointwise estimates for nonnegative solution of the equation (1).

In the work [2] there was considered the double-phase elliptic equation of divergence type

$$-div\left(|\nabla u|^{p-2} \nabla u + a(x)|\nabla u|^{p-2} \nabla u\right) = f(x) \geq 0, \tag{2}$$

in which

$$0 \leq a(x) \in C^{0,\alpha}(\Omega),\ \alpha \in (0,1],\ 1 < p \leq q \leq \min\left(p+\alpha, \frac{np}{n-p}\right),\ q < n. \tag{3}$$

In the paper [2] pointwise estimates for nonnegative solution of double-phase elliptic equations were obtained in terms of nonlinear Wolf potentials:

$$W_{1,p}^{f}(x_0,R) = \sum_{j=0}^{\infty}\left(\rho_j^{p-n} \int_{B_{\rho_j}(x_0)} f dx\right)^{\frac{1}{p-1}},\ \rho_j = \frac{R}{2^j},\ j = 0,1,.... \tag{4}$$

$$W_{1,q}^{f}(x_0,R) = \sum_{j=0}^{\infty}\left(\rho_j^{q-n} \int_{B_{\rho_j}(x_0)} f dx\right)^{\frac{1}{q-1}},\ \rho_j = \frac{R}{2^j},\ j = 0,1,.... \tag{5}$$

In the present paper we generalize the results of papers [1] and [2] and prove analogues results for the quasilinear elliptic equations of general form:

$$-div\left(g(a(x),|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)=f(x)\geq 0.$$

First of all it is necessary to prove iteration lemma. Instead of nonlinear Wolf potentials (4), (5) we will use here the potential

$$W_{1,g}^{f}(x_0,R)=\sum_{j=0}^{\infty}\rho_j\bar{g}\left(\rho_j^{1-n}\int_{B_{\rho_j}(x_0)}fdx\right),\ \rho_j=\frac{R}{2^j},\ j=0,1,.... \tag{6}$$

where the function $\bar{g}$ is inverse function for g.

Let us note that equations under consideration were studied in first by V. V. Zhikov [4-5]. It has been shown that these equations can serve as models of strongly anisotropic materials and can describe the Lavrentiev's phenomena. Moreover, in the book [3] M. Ruzicka establishes the application of such type of quasilinear equations (with nonstandard growth conditions) in modeling of electrorheological fluids behavior.

1. Problem statement

In the bounded domain $\Omega\subset R^n$, $n\geq 2$, let us consider the following quasilinear elliptic equation in the general form:

$$-div\left(g(a(x),|\nabla u|)\frac{\nabla u}{|\nabla u|}\right)=f(x)\geq 0. \tag{7}$$

We will assume that function g satisfies the conditions

$$g\in C(R_+),\ \left(\frac{t}{\tau}\right)^{p-1}\leq\frac{g(a(x_0),t)}{g(a(x_0),\tau)}\leq\left(\frac{t}{\tau}\right)^{q-1},\ t\geq\tau>0,\ 1<p\leq q<n, \tag{8}$$

for any interior fixed point $x_0\in\Omega$, $f(x)\in L^1(\Omega)$. It is necessary to introduce the following definitions.

Definition 1. Let $G(a(x),t)=tg(a(x),t)$. Then the function $u\in W^{1,G}(\Omega)$ if and only if

1) function u is weakly differentiable in Ω,
2) function u satisfies the condition

$$\int_{\Omega}G(a(x),|\nabla u|)\,dx<\infty.$$

Definition 2. The function u is called a weak solution to the equation (7), if

1) $u\in W^{1,G}(\Omega)$

and

2) function u satisfies the integral identity $\int_{\Omega}g(a(x),|\nabla u|)\nabla\varphi\, dx=\int_{\Omega}f\,\varphi\, dx,$

for all $\varphi\in \overset{0}{W}{}^{1,G}$.

2. Iteration lemma

The main result of the present paper is the following iteration lemma for equation (7), which is generalization of obtained earlier analogs for equations (1) and (2).

First of all let us introduce the following notations.

Let $x_0 \in \Omega$ be such that $B_r(x_0) \subset \Omega$.. We set $r_j = \frac{\rho}{2^j}, B_j = B_{r_j}(x_0), j = 0,1,2,....$

and

$$A_j(l) := r_j^{-n} g^{-1-\lambda_0}\left(a(x_0), \frac{l-l_j}{r_j}\right) \int\limits_{B_j \cap \{u > l_j\}} g^{1+\lambda_0}\left(a(x_0), \frac{u-l_j}{r_j}\right) \xi_j^{m-q} dx. \quad (9)$$

We put $l_0 = 0$ and every next $l_{j+1}, j \geq 0$ we define from the condition

$$A_j(l_{j+1}) = k,$$

if $l_j < l \leq \|u\|_{L^\infty(\Omega)}$. For the case $l_j \geq \|u\|_{L^\infty(\Omega)}$ we set $A_j(l) = 0$. Here $k \in (0,1)$ is some fixed positive number, depending only on $p, q, n, [a]_{C^{0,\alpha}(\Omega)}$, $\|u\|^{q-p}_{L^\infty(\Omega)}, a(x_0)$ (if $a(x_0) \neq 0$).

We note

$$\delta_j(l) = l - l_j, \delta_j = \delta_j(l_{j+1}) = l_{j+1} - l_j, L_j = B_j \cap \{u > l_j\},$$

$$\xi_j \in C_0^\infty(B_j),\ 0 \leq \xi_j \leq 1,\ \xi_j(x) \equiv 1,\ \forall x \in B_{j+1}, |\nabla \xi_j| \leq \frac{2}{r_j}.$$

The following statement is a basis of Kilpelainen-Maly technique [1], generalized for equations (7).

Lemma. Let conditions (3) and (8) be fulfilled, $f(x) \in L^1(\Omega)$ and $u \in W^{1,G}(\Omega)$ is a weak nonnegative solution to the Eq. (7). Then for every $j \geq 1$ the next inequalities hold.

$$\delta_j \leq \frac{1}{2}\delta_{j-1} + \gamma r_j \bar{g}\left(r_j^{1-n} \int\limits_{B_j} f dx\right). \quad (10)$$

Proof. We assume $\delta_j \geq \frac{1}{2}\delta_{j-1}$ for every fixed $j \geq 1$. Otherwise (10) is evident. First of all let us prove that

$$|L_j| \leq 2^n r_j^n k. \quad (11)$$

Indeed, for $x \in L_j, \xi_{j-1} = 1$ and $\frac{u - l_j - 1}{\delta_{j-1}} = 1 + \frac{u - l_j}{\delta_{j-1}} \geq 1$ the next inequality is true

$$|L_j| = g^{-1-\lambda_0}\left(a(x_0), \frac{\delta_{j-1}}{r_{j-1}}\right) \int\limits_{L_j} g^{1+\lambda_0}\left(a(x_0), \frac{\delta_{j-1}}{r_{j-1}}\right) \xi_{j-1}^{m-q} dx \leq$$

$$\leq g^{-1-\lambda_0}\left(a(x_0), \frac{\delta_{j-1}}{r_{j-1}}\right) \int\limits_{L_j} g^{1+\lambda_0}\left(a(x_0), \frac{u - l_{j-1}}{r_{j-1}}\right) \xi_{j-1}^{m-q} dx = r_{j-1}^n k.$$

Since $A_j(l_{j+1}) = k$ and due to (9), the last inequality proves (11).

Let us estimate the terms in the right hand side of (9) for $l = l_{j+1}$. We decompose $L_j = L_j' \cup L_j''$. Here $L_j' := \left\{x \in L_j : \frac{u - l_j}{\delta_j} < \varepsilon\right\}$. Small parameter $\varepsilon > 0$ will be determined later.

Using conditions (8) and (11) we have:

$$\int_{L'_j} g^{1+\lambda_0}\left(a(x_0),\frac{u-l_j}{r_j}\right)\xi_j^{m-q}dx \le \varepsilon^{(1+\lambda_0)(p-1)} g^{1+\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right)\left|L_j\right| \le$$

$$\le 2^n \varepsilon^{(1+\lambda_0)(p-1)} r_j^n g^{1+\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right)k. \qquad (12)$$

Let us choose λ by such a way: $0<\lambda<\dfrac{p-1}{(q-1)(n-1)+n}$. For all $x\in L''_j$ the following evident estimate is true

$$g^{1+\lambda}\left(a(x_0),\frac{u-l_j}{r_j}\right) \le \gamma(\varepsilon) g^{-\frac{1}{n-1}+\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right) g^{\frac{n}{n-1}}\left(a(x_0),\frac{u-l_j}{r_j}\right)\left(1+\frac{u-l_j}{r_j}\right)^{-\lambda\frac{n}{n-1}}.$$

Moreover, the estimate

$$\int_L \left|\nabla(\omega\xi^m)\right|dx \le \gamma r^{-1} g^{-\lambda_0}\left(a(x_0),\frac{\delta}{r}\right)\int_L g^{1+\lambda_0}\left(a(x_0),\frac{\delta}{r}\left(1+\frac{u-l}{\delta}\right)\right)\xi^{m-q}dx + \gamma \int_{B_r(x_0)} |f|dx$$

is also fulfilled. From assumption $\delta_j \ge \frac{1}{2}\delta_{j-1}$, (9), (11) and (12) it follows

$$\int_{L''_j} g^{1+\lambda_0}\left(a(x_0),\frac{u-l_j}{r_j}\right)\xi_j^{m-q}dx \le \gamma(\varepsilon) g^{-\frac{1}{n}+\lambda_0\frac{n-1}{n}}\left(a(x_0),\frac{\delta_j}{r_j}\right)\times$$

$$\times\left(\int_{L''_j} \frac{g^{\frac{n}{n-1}}\left(a(x_0),\frac{\delta_j}{r_j}\upsilon_j^{\frac{p}{p-1-\lambda}}\right)}{\left(1+\frac{u-l_j}{\delta_j}\right)^{\lambda\frac{n}{n-1}}}\xi_j^{m-q}dx\right)^{\frac{n-1}{n}} \left(\int_{L''_j} g^{1+\lambda_0}\left(a(x_0),\frac{u-l_j}{r_j}\right)\xi_j^{m-q}dx\right)^{\frac{1}{n}} \le$$

$$\le \gamma(\varepsilon) r_j k^{\frac{1}{n}} g^{\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right)\left(\int_{L_j}\left(\omega_j \xi_j^{(m-q)\frac{n-1}{n}}\right)^{\frac{n}{n-1}}dx\right)^{\frac{n-1}{n}} \le$$

$$\le \gamma(\varepsilon) r_j k^{\frac{1}{n}} g^{\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right)\int_{L_j}\left|\nabla\left(\omega_j\xi_j^{(m-q)\frac{n-1}{n}}\right)\right|dx \le \gamma(\varepsilon) r_j k^{\frac{1}{n}} g^{\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right)\times$$

$$\times\left(g^{-\lambda}\left(a(x_0),\frac{\delta_j}{r_j}\right)\int_{L_j} g^{1+\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\left(1+\frac{u-l_j}{\delta_j}\right)\right)\xi_j^{(m-q)\frac{n-1}{n}-q}dx + r_j\int_{B_j}|f|dx\right). \qquad (13)$$

Let us choose m from condition $(m-q)\dfrac{n-1}{n}-q\ge 1$, then

$$\int_{L_j} g^{1+\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\left(1+\frac{u-l_j}{\delta_j}\right)\right)\xi_j dx \le \gamma(\varepsilon) r_j^n g^{1+\lambda_0}\left(a(x_0),\frac{\delta_j}{r_j}\right)k. \qquad (14)$$

Taking into account (12), (13) and (14), we arrive at

$$k \le 2^n \varepsilon^{(1+\lambda)(p-1)} k + \gamma(\varepsilon) k^{\frac{1}{n}}\left(k + g^{-1}\left(a(x_0),\frac{\delta_j}{r_j}\right) r_j^{1-n}\int_{B_j} f dx\right). \qquad (15)$$

We choose ε sufficiently small, $2^n \varepsilon^{(1+\lambda_0)(p-1)} = \frac{1}{4}$, and further we choose $k = k(\varepsilon)$: $\gamma(\varepsilon) k^{\frac{1}{n}} = \frac{1}{4}$. So, estimate (10) follows from $A_j(l_{j+1}) = k$, (9) and (15). Lemma is proved.

3. Application of Iteration lemma

Proved lemma can be applied for obtaining Harnack-type inequality for nonnegative weak solutions to the Eq. (7).

Let us sum up inequalities (10) for $j = 1,2...,J-1$.

$$l_J \le \gamma\delta_0 + \gamma W^f_{1,g}(x_0, 2\rho). \quad (16)$$

Here $W^f_{1,g}(x_0,2\rho)$ is nonlinear Wolf potential defined by (6). From definition l_1 we have $\delta_0 < \infty$, then sequence $\{l_j\}_{j\in N}$ is convergent and $\delta_j \to 0$, if $j \to \infty$. Passing to the limit $J \to \infty$ in (16), we obtain $\frac{1}{r_j^n}\int_{B_j} (u-l)_+^{(1+\lambda_0)(p-1)} \le \gamma\delta_j^{(1+\lambda_0)(p-1)} \to 0, j \to \infty$, where $l := \lim_{j\to\infty} l_j$. We choose x_0 as a Lebesque point of the function $(u-l)_+^{(1+\lambda_0)(p-1)}$. Using (8) we arrive at $u(x_0) \le l$. If $u(x_0) \ge 2\gamma W^f_{1,g}(x_0,2\rho)$, (16) implies

$$g^{1+\lambda_0}\left(a(x_0), \frac{u(x_0)}{\rho}\right) \le \gamma\rho^{-n} \int_{B_\rho(x_0)} g^{1+\lambda_0}\left(a(x_0), \frac{u}{\rho}\right) dx. \quad (17)$$

Here $0 < \lambda_0 < \frac{1}{n-1}$. So, we have shown that iteration lemma can be applied for proving Harnack-type inequality $u(x_0) \le \gamma W^f_{1,g}(x_0,2\rho)$ (or (17)) for nonnegative weak solution to the equation (7).

Conclusions

In the present paper it has been proved iteration lemma for general case of quasilinear elliptic equation (7). It has been shown the application of it for proving Harnack-type inequalities. Our statements generalize the results obtained earlier for equations (1) and (2) (see [1], [2]).

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$$-div\left(g(a(x),|\nabla u|)\frac{\nabla u}{|\nabla u|} \right) = f(x) \geq 0$$

доведено ітераційну лему. Як і у випадку оператора p-Лапласа, для якого подібна лема була вперше встановлена Kilpelainen і Maly, отриманий результат служить основним інструментом для подальшого дослідження квазілінійних еліптичних рівнянь такого типу. За допомогою цієї леми доведено нерівність типу Гарнака для рівнянь, що розглядаються, в термінах нелінійних потенціалів Вольфа.

Ключові слова: квазілінійні еліптичні рівняння, ітераційна техніка, рівняння в частинних похідних другого порядку, оператор p-Лапласа, потенціали Вольфа, двофазні рівняння, поточкові оцінки.

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